TUTORIUM 04.05.2010

Integration

Ablauf: Tutorium M2 (4.5.10)

$$F(x) := \int_0^x \frac{dt}{\cos(t)} = -\int_1^{\cos(x)} \frac{d\tau}{\tau \sin(t)}$$

BÄH!

$$\boxed{\tau := \cos(t) \qquad \frac{d\tau}{dt} = -\sin(t)}$$

Motivation: Hyperbel (cos(t)) ???

Lösung mit Hilfe der Halbwinkelmethode

$$F(x) := \int_0^x \frac{dt}{\cos(t)} = \int_0^{\tan(\frac{x}{2})} \frac{\cancel{1+\tau^2}}{1-\tau^2} \cdot \frac{2}{\cancel{1+\tau^2}} \, d\tau$$

$$\boxed{\tau = \tan\left(\frac{t}{2}\right), \quad \cos(t) = \frac{1-\tau^2}{1+\tau^2}, \quad dt = \frac{2}{1+\tau^2}\, d\tau}$$

$$= \int_0^{\tan(\frac{x}{2})} \underbrace{\frac{2}{1-\tau^2}}_{\text{PBZ}} \, d\tau$$

$$\boxed{\begin{aligned} \text{Integrand} &= \frac{-2}{(\tau-1)(\tau+1)} = \frac{A}{\tau-1} + \frac{B}{\tau+1} \\ &= \frac{-1}{\tau-1} + \frac{1}{\tau+1} \end{aligned}}$$

$$= -\int_0^{\tan(\frac{x}{2})} \frac{1}{\tau-1} \, d\tau + \int_0^{\tan(\frac{x}{2})} \frac{1}{\tau+1} \, d\tau$$

jeweils Typ!

$$\int \frac{f'(t)}{f(t)} \, dt$$

$$= -\ln(|\tau-1|)\Big|_0^{\tan(\frac{x}{2})} + \ln(|\tau+1|)\Big|_0^{\tan(\frac{x}{2})}$$ ②

$$= -\ln(|\tan(\tfrac{x}{2})-1|) + \cancel{\ln(|-1|)}$$
$$+ \ln(|\tan(\tfrac{x}{2})+1|) - \cancel{\ln(|1|)}$$

$$= \ln\left(\left|\frac{\tan(\frac{x}{2})+1}{\tan(\frac{x}{2})-1}\right|\right) = F(x)$$

$\boxed{\text{Probe}}$ selber machen!!!

$$F'(x) = \frac{\cancel{\tan(\frac{x}{2})-1}}{\tan(\frac{x}{2})+1} \cdot \frac{(\tan(\frac{x}{2})-1)\frac{1}{\cos^2(\frac{x}{2})}\cdot\frac{1}{2} - (\tan(\frac{x}{2})+1)\frac{1}{\cos^2(\frac{x}{2})}\cdot\frac{1}{2}}{(\tan(\frac{x}{2})-1)^{\cancel{2}}}$$

$$= \frac{1}{2}\cdot\frac{1}{\cos^2(\frac{x}{2})}\cdot\frac{\cancel{\tan(\frac{x}{2})}-1-(\cancel{\tan(\frac{x}{2})}+1)}{\underbrace{\tan^2(\tfrac{x}{2})-1}}$$

$$\text{Nenner} = \frac{\sin^2(\frac{x}{2})-\cos^2(\frac{x}{2})}{\cos^2(\frac{x}{2})} = \frac{-\cos(2\cdot\frac{x}{2})}{\cos^2(\frac{x}{2})}$$

$$= \frac{1}{\cancel{2}}\cdot\frac{1}{\cancel{\cos^2(\frac{x}{2})}}\cdot\frac{+\cancel{2}\cdot\cancel{\cos^2(\frac{x}{2})}}{+\cos(x)} = \frac{1}{\cos(x)}$$

WEITERE BEISPIELE

Beispiel 1

$$F(x) := \int_0^x \frac{1}{\cosh(t)}\,dt = \int_0^x \frac{2}{e^t + e^{-t}}\,dt$$

$$\boxed{\tau := e^t \Rightarrow \frac{d\tau}{dt} = e^t = \tau \Leftrightarrow dt = \frac{d\tau}{\tau}}$$

$$= 2 \cdot \int_1^{e^x} \frac{1}{\tau + \tau^{-1}} \cdot \frac{d\tau}{\tau} = 2 \cdot \int_1^{e^x} \frac{1}{1 + \tau^2}\,d\tau$$

$$= 2\arctan(\tau)\Big|_1^{e^x} = 2\arctan(e^x) - 2\arctan(1)$$

Beispiel ②

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Idee

$$r^2\cos^2(x) + r^2\sin^2(x) = 1\cdot r^2$$

$$\Leftrightarrow \quad r\cos(x) = \sqrt{r^2 - r^2\sin^2(x)}$$

$$= \sqrt{r^2 - (\underbrace{r\sin(x)}_{t})^2}$$

$$F(x) := \int_0^x \frac{1}{\sqrt{9 - t^2}}\,dt$$

3^2

$\boxed{1}$ $t := 3\sin(\tau)$

$\boxed{2}$ $\frac{dt}{d\tau} = 3\cos(\tau)$

$\Leftrightarrow dt = 3\cos(\tau)\,d\tau$

$\boxed{3}$ $3\cos(\tau) = \sqrt{3^2 - 3^2\sin^2(\tau)}$

nach obiger Regel

$\boxed{4}$ für die Transformation der Grenzen

$\tau = \arcsin\left(\frac{t}{3}\right)$

$\boxed{1-4}$

$$= \int_0^{\arcsin(\frac{x}{3})} \frac{3\cos(\tau)}{3\cos(\tau)}\,d\tau = \int_0^{\arcsin(\frac{x}{3})} 1\,d\tau = \tau\Big|_0^{\arcsin(\frac{x}{3})}$$

$$= \arcsin\left(\frac{x}{3}\right) = F(x)$$

Probe

$$F'(x) = \frac{1}{\sqrt{1-(\frac{x}{3})^2}}\cdot\frac{1}{3} = \frac{1}{\sqrt{9}\cdot\sqrt{1-(\frac{x}{3})^2}}$$

$$= \frac{1}{\sqrt{9 - 9(\frac{x}{3})^2}} = \frac{1}{\sqrt{9 - \cancel{3}^2(\frac{x}{\cancel{3}})^2}} = \frac{1}{\sqrt{9 - x^2}}$$

Beispiel 3

$$F(x) := \int_1^x \frac{1}{\sqrt{t^2 - \underbrace{9}_{3^2}}}\, dt$$

Idee

$$r^2(\cosh^2(x) - \sinh^2(x)) = 1\, r^2$$

$$\Leftrightarrow r\sinh(x) = \sqrt{\underbrace{r^2\cosh^2(x)}_{t^2} - r^2}$$

$$\boxed{\begin{aligned} &t := 3\cosh(\tau) \qquad dt = 3\sinh(\tau)\, d\tau \\ &\sqrt{3^2\cosh^2(\tau) - 3^2} = 3\sinh(\tau) \\ &\tau = \operatorname{arcosh}\left(\tfrac{t}{3}\right) \end{aligned}}$$

$$= \int_0^{\operatorname{arcosh}(\frac{x}{3})} \frac{3\sinh(\tau)}{3\sinh(\tau)}\, d\tau = \tau \Big|_0^{\operatorname{arcosh}(\frac{x}{3})}$$

$$= \operatorname{arcosh}\left(\frac{x}{3}\right) = F(x)$$

Probe

wie zuvor!

$$F'(x) = \frac{1}{\sqrt{\left(\frac{x}{3}\right)^2 - 1}} \cdot \frac{1}{3} = \frac{1}{\sqrt{x^2 - 9}}$$